Sec 3.3 Reduced echelon form

Last time: row operations to reduce matrix to (row-) echelon form

$$\begin{bmatrix} 1 & 3 & 9 \\ 2 & 1 & 8 \end{bmatrix} \xrightarrow{(R_2 \to R_2 + (-2)R_1)} \begin{bmatrix} \boxed{1} & 3 & 9 \\ 0 & \boxed{-5} & -10 \end{bmatrix}$$

#s under pivots are 0 pivots

Reduced row echelon form: Echelon form, AND
(RREF)

- all pivots = 1.
- all #'s below & above pivot are 0

$$\begin{bmatrix} 1 & 3 & 9 \\ 0 & -5 & -10 \end{bmatrix} \quad (EF)$$

$$(R_2 \to -\tfrac{1}{5}R_2) \begin{bmatrix} 1 & 3 & 9 \\ 0 & 1 & 2 \end{bmatrix} \quad (EF)$$

$$(R_1 \to R_1 - 3R_2) \begin{bmatrix} \boxed{1} & 0 & 3 \\ 0 & \boxed{1} & 2 \end{bmatrix} \quad (RREF)$$

★ Matrices can be row reduced to many different EF's, but only one RREF!

$$(R_2 \leftrightarrow R_1) \begin{bmatrix} 2 & 1 & 8 \\ 1 & 3 & 9 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 1 & 8 \\ 0 & 2.5 & 5 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 2 & 1 & 8 \\ 0 & 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 0 & 6 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix}$$

Ex
$$\begin{cases} x_1 + 2x_2 + x_3 = 4 \\ 3x_1 + 8x_2 + 7x_3 = 20 \\ 2x_1 + 7x_2 + 9x_3 = 23 \end{cases}$$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 3 & 8 & 7 & 20 \\ 2 & 7 & 9 & 23 \end{array}\right]$$

$\begin{pmatrix} R_2 \rightarrow R_2 - 3R_1 \\ R_3 \rightarrow R_3 - 2R_1 \end{pmatrix}$
$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 2 & 4 & 8 \\ 0 & 3 & 7 & 15 \end{array}\right]$$

$\left(R_2 \rightarrow \frac{1}{2}R_2\right)$
$$\left[\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 4 \\ 0 & \boxed{1} & 2 & 4 \\ 0 & \boxed{3} & 7 & 15 \end{array}\right]$$

$(R_3 \rightarrow R_3 - 3R_2)$
$$\begin{array}{ccc} x_1 & x_2 & x_3 \end{array}$$
$$\left[\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 4 \\ 0 & \boxed{1} & 2 & 4 \\ 0 & 0 & \boxed{1} & 3 \end{array}\right] \quad (EF)$$

same as
$$\begin{cases} x_1 + 2x_2 + x_3 = 4 \\ x_2 + 2x_3 = 4 \\ \underline{x_3 = 3} \end{cases}$$

$\underline{x_2} = 4 - 2x_3 = \underline{-2}$

$\underline{x_1} = 4 - 2x_2 - x_3 = 4 + 4 - 3 = 5$

solution vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix} = (5, -2, 3)$

★ RREF doesn't require "back-substitution":

continue with

$$\begin{pmatrix} R_1 \to R_1 - R_3 \\ R_2 \to R_2 - 2R_3 \end{pmatrix} \left[\begin{array}{ccc|c} \boxed{1} & 2 & 0 & 1 \\ 0 & \boxed{1} & 0 & -2 \\ 0 & 0 & \boxed{1} & 3 \end{array}\right]$$

$$(R_1 \to R_1 + (-2)R_2) \left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 5 \\ 0 & \boxed{1} & 0 & -2 \\ 0 & 0 & \boxed{1} & 3 \end{array}\right] \text{(RREF)}$$

Now "read off" solution
$$\begin{cases} x_1 = 5 \\ x_2 = -2 \\ x_3 = 3 \end{cases}$$

sol. vector form $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix} \left(= (5, -2, 3) \right)$

Echelon form $\longleftrightarrow$ Gaussian Elimination

Reduced EF $\longleftrightarrow$ Gauss – Jordan " "

<u>Ex</u>
$$\begin{cases} x_1 + 3x_2 - 15x_3 + 7x_4 = 0 \\ x_1 + 4x_2 - 19x_3 + 10x_4 = 0 \\ 2x_1 + 5x_2 - 26x_4 + 11x_4 = 0 \end{cases}$$

all zeros
→ "<u>homogeneous syst. of eqs</u>"

unrelated to $f(y/x)$ ...

homogeneous systems always have "<u>trivial solution</u>" (all variables = 0)

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

possibly more.

$$\left[\begin{array}{cccc|c} 1 & 3 & -15 & 7 & 0 \\ 1 & 4 & -19 & 10 & 0 \\ 2 & 5 & -26 & 11 & 0 \end{array}\right]$$

$$\begin{pmatrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{pmatrix} \left[\begin{array}{cccc|c} 1 & 3 & -15 & 7 & 0 \\ 0 & 1 & -4 & 3 & 0 \\ 0 & -1 & 4 & -3 & 0 \end{array}\right]$$

$$\begin{pmatrix} R_1 \to R_1 - 3R_2 \\ R_3 \to R_3 + R_1 \end{pmatrix} \left[\begin{array}{cccc|c} \boxed{1} & 0 & -3 & -2 & 0 \\ 0 & \boxed{1} & -4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(RREF)

$$\underbrace{x_1 \quad x_2}_{\text{pivot vars}} \qquad \underbrace{x_3 \quad x_4}_{\text{free vars}}$$

set $\underline{x_3 = s}$, $\underline{x_4 = t}$, free params

$R_1 \Rightarrow \underline{x_1 = 0 + 3s + 2t}$

$R_2 \Rightarrow \underline{x_2 = 0 + 4s - 3t}$

<u>solution vectors</u>

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3s + 2t \\ 4s - 3t \\ s \\ t \end{bmatrix}$$

$$= s \begin{bmatrix} 3 \\ 4 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$